UDC 539.3

ON THE STATE OF STRESS OF A STRIP WEAKENED BY IDENTICAL CIRCULAR TRANSVERSE OPENINGS

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The title problem is solved using the Sherman [1] method which makes it possible to reduce the problem in question to an auxilliary problem for a solid strip, a solution for which is obtained with help of the Fourier transform. The solution of the problem is reduced, in the last stage, to an infinite system of linear algebraic equations, and the system is at least quasiregular at arbitrarily small distances from the boundary. A numerical analysis is also given for the following three variants of loading of the strip: for longitudinal tension, uniform transverse tension and for uniform pressure along the hole contours.

Let us consider a strip of width $2a$, weakened by two holes of equal radii R, symmetrically distributed in transverse direction, their centers at the distance $2c$ from each other (Fig. 1). The strip is acted upon by uniform tensile forces T_x and

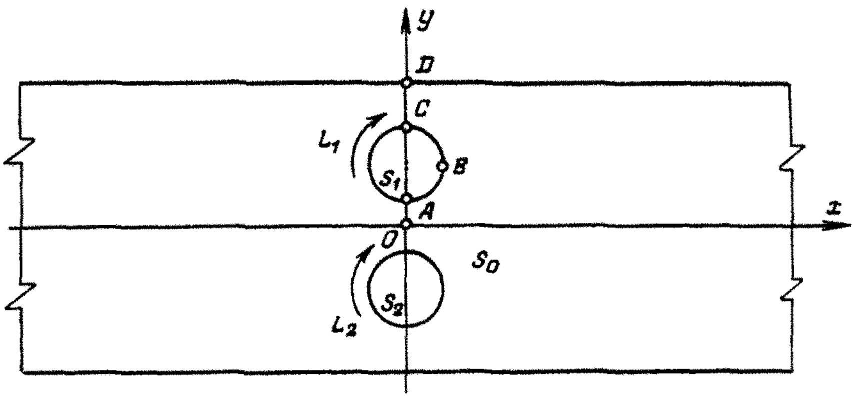

Fig. 1

T_y in the longitudinal and transverse direction respectively. We denote by S_0 the triply connected region of the strip, and by S_1, S_2 the regions contained within the circles L_1 and L_2. The area of the solid strip is $S = S_0 + S_1 + S_2$. The unknown stresses are conveniently written in the form

$$X_x^{(1)} = T_x + X_x, \quad Y_y^{(1)} = T_y + Y_y, \quad X_y^{(1)} = X_y$$

where X_x, Y_y and X_y represent the result of the perturbation caused by the presence of holes. The Kolosov—Muskhelishvili potentials corresponding to the stresses X_x, Y_y and X_y are denoted by $\varphi_1(z)$ and $\psi_1(z)$. The boundary conditions on the longitudinal boundaries are obvious for the additional stresses X_x, Y_y and X_y, and on the circles L_j $(j = 1, 2)$ they have the form

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = 2h_1(t - b_j) + \frac{2h_2R^2}{t - b_j} + 2C_j$$

$$(h_1 = -(T_x + T_y)/4,\ h_2 = (T_x - T_y)/4,\ b_1 = -b_2 = ic)$$

where C_j is a constant not affecting the state of stress.

Following [1], we introduce a new unknown function

$$2\omega(t) = \varphi_1(t) - t\,\overline{\varphi_1'(t)} - \overline{\psi_1(t)} \quad \text{on } L_j, \quad j = 1, 2$$

and this enables us to construct the functions $\varphi(z)$ and $\psi(z)$ analytic in the region S, i.e. in the solid strip. In the region S_0 these functions are given by the formulas

$$\varphi(z) = \varphi_1(z) - \sum_{j=1}^{2}\left(\varphi_j^*(z) + \frac{h_2R^2}{z - b_j}\right) \tag{1}$$

$$\psi(z) = \psi_1(z) - \sum_{j=1}^{2}\left[\psi_j^*(z) + h_2R\sum_{k=2}^{3}\lambda_{jk}\left(\frac{R}{z - b_j}\right)^k\right]$$

$$\varphi_j^*(z) = \frac{1}{2\pi i}\int_{L_j}\frac{\omega(t)\,dt}{t - z}$$

$$\psi_j^*(z) = -\frac{1}{2\pi i}\int_{L_j}\frac{\overline{\omega(t)} + \bar{t}\omega'(t)}{t - z}\,dt$$

$$\lambda_{j2} = -b_j/R, \quad \lambda_{j3} = 1$$

The integrals in these and in the following formulas are taken in the clockwise direction.

Let us write the functions $\varphi_j^*(z)$ and $\psi_j^*(z)$ for z lying outside L_j, in the form of series

$$\varphi_j^*(z) = -\sum_{k=0}^{\infty}\alpha_{kj}\left(\frac{R}{z - b_j}\right)^{k+1}, \quad \psi_j^*(z) = \sum_{k=0}^{\infty}\beta_{kj}^{(1)}\left(\frac{R}{z - b_j}\right)^{k+1}. \tag{2}$$

$$\beta_{0j}^{(1)} = -\beta_{0j} - \overline{\beta}_{0j}$$

$$\beta_{kj}^{(1)} = -\overline{\beta}_{kj} + k\frac{b_j}{R}\alpha_{k-1,j} - (k-1)\alpha_{k-2,j}, \quad k \geqslant 1$$

$$\alpha_{kj} = \frac{1}{2\pi iR^{k+1}}\int_{L_j}\omega(t)(t - b_j)^k\,dt, \quad \beta_{kj} = \frac{1}{2\pi iR^{k+1}}\int_{L_j}\omega(t)\overline{(t - b_j)^k\,dt}$$

Assuming that α_{kj} and β_{kj} are the Fourier coefficients of the function $\omega(t)$ on L_1 and L_2 and taking into account the symmetric character of the state of stress relative to both axes, we have $\alpha_{k2} = \bar{\alpha}_{k1}, \beta_{k2} = \bar{\beta}_{k1}$. When $k = 0, 2, 4, \ldots$,

α_{kj} and β_{kj} are real, and for $k = 1, 3, 5, \ldots$ they are purely imaginary. The formulas (1) and (2) yield expressions for the potentials $\varphi_1(z)$ and $\psi_1(z)$ sought, in terms of the functions $\varphi(t)$ and $\psi(t)$ (which have the corresponding state of stress $X_x^{(2)}$, $Y_y^{(2)}$, $X_y^{(2)}$)

$$\varphi_1(z) = \varphi(z) - \sum_{j=1}^{2}\sum_{k=0}^{\infty} \alpha_{kj}^{**}\left(\frac{R}{z-b_j}\right)^{k+1} \tag{3}$$

$$\psi_1(z) = \psi(z) - \sum_{j=1}^{2}\sum_{k=0}^{\infty} \beta_{kj}^{(2)}\left(\frac{R}{z-b_j}\right)^{k+1}$$

$$\alpha_{0j}^{**} = \alpha_{0j} - h_2 R, \quad \beta_{0j}^{**} = 2\beta_{0j}; \quad \alpha_{kj}^{**} = \alpha_{kj}, \quad \beta_{kj}^{**} = \beta_{kj}, \quad k \geqslant 1$$

$$\beta_{0j}^{(2)} = \beta_{0j}^{**}, \quad \beta_{kj}^{(2)} = \bar{\beta}_{kj}^{**} - k\frac{b_j}{R}\alpha_{k-1,j}^{**} + (k-1)\alpha_{k-2,j}^{**}, \quad k \geqslant 1$$

Taking into account what has been said so far, we put (α_{kj}^* and β_{kj}^* are real quantities)

$$\alpha_{kj}^{**} = \alpha_{kj}^*, \quad \beta_{kj}^{**} = \beta_{kj}^*, \quad k = 0, 2, 4, \ldots$$
$$\alpha_{kj}^{**} = i\alpha_{kj}^*, \quad \beta_{kj}^{**} = i\beta_{kj}^*, \quad k = 1, 3, 5, \ldots$$

Using relations (3) we can [1 —3] reduce the solution of the problem in question to a solution of an intermediate problem for the region S. The boundary conditions for the last problem are:

$$Y_k^{(2)} - iX_y^{(2)} = -\sum_{j=1}^{2}\sum_{k=0}^{\infty}\frac{k+1}{R}\left\{\alpha_{kj}^{**}\left(\frac{R}{t-b_j}\right)^{k+2} + \gamma_{kj}^{**}\left(\frac{R}{\overline{t-b_j}}\right)^{k+2} - \right. \tag{4}$$
$$\left.(k+2)\bar{\alpha}_{kj}^{**}\frac{t}{R}\left(\frac{R}{\overline{t-b_j}}\right)^{k+3}\right\}, \quad t = x \pm ia$$

$$\gamma_{0j}^{**} = \bar{\alpha}_{0j}^{**} + \beta_{0j}^{**}, \quad \gamma_{kj}^{**} = \bar{\alpha}_{kj}^{**} + k\frac{b_j}{R}\bar{\alpha}_{k-1,j}^{**} + (k-1)\bar{\alpha}_{k-2,j}^{**} + \beta_{kj}^{**}, \quad k \geqslant 1$$

and the solution obtained using the integral Fourier transforms, has the form

$$\varphi(z) = -i\int_{-\infty}^{\infty} H_1(\mu)\, e^{-iz\mu/a}\frac{d\mu}{\mu} \tag{5}$$

$$\psi(z) = i\int_{-\infty}^{\infty}\left[\left(1 - iz\frac{\mu}{a}\right)H_1(\mu) + 2H_2(\mu)\right]e^{-iz\mu/a}\frac{d\mu}{\mu}$$

$$H_1(\mu) = \sum_{j=0}^{\infty} T_j(\mu)\{[(a_j(\mu) + 2\gamma(\mu))\Gamma_{1,j}(\mu) + 2\varepsilon_2\mu\Gamma_{3,j}(\mu)]\alpha_j^* - \Gamma_{2,j}(\mu)\beta_j^*\}$$

$$H_2(\mu) = \sum_{j=0}^{\infty} T_j(\mu)\{[(2\mu^2 - a_j(\mu)b(\mu))\Gamma_{1,j}(\mu) - 2\varepsilon_2\mu b(\mu)\Gamma_{3,j}(\mu)]\alpha_j^* +$$
$$b(\mu)\Gamma_{2,j}(\mu)\beta_j^*\}, \quad T_j(\mu) = \frac{\varepsilon_1^{j+1}}{j!}\frac{\mu^{j+1}}{2\mu + \operatorname{sh} 2\mu}$$

$$\left.\begin{matrix}\Gamma_{1,j}(\mu)\\ \Gamma_{2,j}(\mu)\end{matrix}\right\} = \cos j\frac{\pi}{2}\operatorname{ch}\varepsilon_2\mu \pm \sin j\frac{\pi}{2}\operatorname{sh}\varepsilon_2\mu, \quad \alpha_j(\mu) = j + \frac{(\varepsilon_1\mu)^2}{j+2}$$

$$\Gamma_{3,j}(\mu) = \sin j\frac{\pi}{2}\operatorname{ch}\varepsilon_2\mu + \cos j\frac{\pi}{2}\operatorname{sh}\varepsilon_2\mu, \quad b(\mu) = 1 - \gamma(\mu)$$

$$2\gamma(\mu) = 1 - 2\mu + e^{-2\mu}, \quad \varepsilon_1 = R/a, \quad \varepsilon_2 = c/a$$

It should be noted that $H_1(\mu)$ and $H_2(\mu)$ are symmetric functions, and the expressions given here are for $\mu > 0$ only.

The only unknown quantities left are $\alpha_j^* \equiv \alpha_{j1}^*$ and $\beta_j^* \equiv \beta_{j1}^*$. To find them, we require an infinite system of linear algebraic equations. This can be constructed e.g. by first obtaining [1] an integral equation (for $\omega(t)$) with a degenerate kernel

$$\omega(t) = \varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + \sum_{k=0}^{\infty}\left[\Omega_k^{(1)}\left(\frac{t-b_1}{R}\right)^k + \Omega_k^{(2)}\left(\frac{R}{t-b_1}\right)^k\right] + \qquad (6)$$

$\alpha_{-1,1}$ on L_1

$$\Omega_1^{(1)} = \frac{\beta_0^*}{2}, \quad \Omega_k^{(1)} = (-1)^{k+1}\sum_{n=0}^{\infty}\overline{\alpha}_{n,1}^{**}C_{n+k}^k\left(\frac{\varepsilon_3}{2i}\right)^{n+k+1}, \quad k \neq 1$$

$$\Omega_k^{(2)} = \delta_k h_2 R + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{\varepsilon_3}{2i}\right)^{n+k+1}\left\{(n+1)\alpha_{n,1}^{**}\left[C_{n+k+1}^k - \left(\frac{\varepsilon_3}{2}\right)^2 C_{n+k+3}^{k+1}\right] + \overline{\beta}_{n,1}^{**}C_{n+k}^k\right\}$$

$$C_m^n = \frac{m!}{n!\,(m-n)!}, \quad \varepsilon_3 = \frac{R}{c}; \quad \delta_1 = 1, \quad \delta_k = 0, \quad k \neq 1$$

The functions $\varphi(t)$ and $\psi(t)$ can be found using formulas (5). Solving the integral equation (6), we arrive at the required system

$$\sum_{j=1}^{\infty} a_{kj}x_j = g_k, \quad k = 1, 2, 3, \ldots \qquad (7)$$

$$x_{2j+1} = \alpha_j^*, \quad x_{2j+2} = \beta_j^*; \quad g_1 = -2h_2R, \quad g_2 = -h_1R, \quad g_k = 0, \quad k \geqslant 3$$

and we write the coefficients a_{kj} as follows ($\delta_{i,j}$ is the Kronecker delta)

$$a_{2m+1,\,2n+1} = \delta_{2m+1,\,2n+1} - \int_0^{\infty} T_{mn}(\mu)\{\Gamma_{mn}^{11}(\mu)[4\mu^2 + 2\gamma(\mu)(2 + a_m(\mu) + a_n(\mu)) + a_m(\mu)a_n(\mu)] + 2\varepsilon_2\mu\,[f_m(\mu)\Gamma_{mn}^{13}(\mu) + f_n(\mu)\Gamma_{mn}^{31}(\mu)] + 4\varepsilon_2^2\mu^2\Gamma_{mn}^{33}(\mu)\}\,d\mu + (n+1)\,\varepsilon_{mn}K(m,n)\left[C_{m+n+2}^{m+1} - \left(\frac{\varepsilon_3}{2}\right)^2 C_{m+n+4}^{m+2}\right] \qquad (8)$$

$$a_{2m+2,\,2n+2} = \delta_{n+1}^{(1)}\delta_{2m+2,\,2n+2} - \int_0^{\infty} T_{mn}(\mu)\,\Gamma_{mn}^{22}(\mu)\,d\mu$$

$$a_{2m+1,\,2n+2} = \int_0^{\infty} T_{mn}(\mu)[f_m(\mu)\Gamma_{mn}^{12}(\mu) + 2\varepsilon_2\mu\Gamma_{mn}^{32}(\mu)]\,d\mu + \tau_{mn}K(m,-n)$$

$$a_{2m+2,\,2n+1} = \int_0^{\infty} T_{mn}(\mu)\,[f_n(\mu)\Gamma_{mn}^{21}(\mu) + 2\varepsilon_2\mu\Gamma_{mn}^{23}(\mu)]\,d\mu + \tau_{mn}K(-m,n)$$

$$T_{mn}(\mu) = \frac{2\varepsilon_1^{m+n+2}}{(m+1)!\,n!}\frac{\mu^{m+n+1}}{2\mu + \operatorname{sh} 2\mu}$$

$$K(m, n) = \cos(m-n)\frac{\pi}{2} - \sin(m+n)\frac{\pi}{2}$$

$$\Gamma_{mn}^{ij}(\mu) = \Gamma_{i,m}(\mu)\Gamma_{j,n}(\mu), \quad \varepsilon_{mn} = (\varepsilon_2/2)^{m+n+2}$$

$$f_j(\mu) = 2\gamma(\mu) + a_j(\mu), \quad \tau_{mn} = \varepsilon_{mn}C_{m+n+1}^{m+1}$$

$$\delta_1^{(1)} = 1/2, \quad \delta_k^{(1)} = 1, \quad k \geqslant 2; \quad m, n = 0, 1, 2, \ldots$$

Relations (8) yield the following expressions which reduce considerably the amount of numerical computations required:

$$\frac{a_{\alpha,\beta}}{a_{\beta,\alpha}} = \frac{n+1}{m+1}; \quad \alpha = 2m+1, \quad 2m+2; \quad \beta = 2n+1, \quad 2n+2$$

Analysing the expressions (8) we find, that the system (7) is at least quasiregular at arbitrarily close distances from the boundary. Solving the system (7) and then computing the stresses with help of the known [4] formulas, we complete the investigation of the problem formulated above.

Numerical analysis, some results of which are quoted below, was carried out for $\varepsilon_1 = 0.1, 0.2, 0.3, 0.4, 0.45$ and $\varepsilon_2 = (1+\varepsilon_1)/3$ (in the latter case all three necks have the same width. We recall that ε_1 and ε_2 are the ratios of the hole radius and the half-distance between the hole centers respectively, to the half-width of the strip). The computations were carried out for three cases of strip loading given at the beginning of this paper. The infinite system (7) was truncated, for the given values of ε_1, to 14, 18, 26, 34 and 40 equations. The longitudinal boundary conditions were satisfied in all cases practically 100%. The absolute error in the worst case of $\varepsilon_1 = 0.45$ was of the order of 10^{-6}. At the hole contours the boundary conditions were also satisfied to high order of accuracy, the error in the worst case of $\varepsilon_1 = 0.45$ being equal to 0.57%.

All computations were carried out in the first quadrant, and some of the results are given in tables. The upper line in each pair of values corresponds to the case of a strip stretched in the longitudinal direction by the force $T_x = \text{const}$, and the lower line to the stretching by the force $T_y = \text{const}$. Multiplying the relevant values by T_x or T_y as appropriate, gives the actual values of the stresses. The Tables can also be used to obtain the corresponding stresses for a strip subjected to a uniform load P along the hole contours. In this case both numbers are added together, the sum reduced by 1, and the result multiplied by P.

Table 1 gives the stresses σ_θ along the hole contours (angle θ is counted from the point A in the direction ABC).

The stress σ_θ at the hole contours was computed using a 15° step. Table 1 quotes the values for a 30° step, and this means that some of the extremal values (from the computations using a narrower step) are missing. We give them here: for $T_y = 0$, $\varepsilon_1 = 0.4$, $\theta = 105°$: $\sigma_\theta = -1.8489\ T_x$, for $T_x = 0$, $\varepsilon_1 = 0.3$, $\theta = 105°$: $\sigma_\theta = 3.8030\ T_y$ and for $T_x = 0$, $\varepsilon_1 = 0.45$, $\theta = 135°$: $\sigma_\theta = 8.7166\ T_y$.

Table 2 gives the values of the stress $X_x^{(1)}$ along the vertical symmetry axis. The points are sampled in the direction $OACD$, using a step $(c - R)/3$ on the

Table 1

ε_1	$\theta°=0$	30	60	90	120	150	180
0.1	3.068	2.079	0.054	−1.006	−0.011	2.054	3.098
	−0.960	−0.021	1.941	3.006	2.051	0.010	−1.029
0.2	3.313	2.412	0.219	−1.108	−0.160	2.196	3.503
	−0.816	−0.205	1.745	3.181	2.401	0.082	−1.255
0.3	4.193	3.138	0.411	−1.331	−0.588	2.333	4.600
	−0.832	−0.652	1.528	3.610	3.321	0.490	−1.956
0.4	7.967	4.507	0.625	−1.610	−1.558	1.812	8.146
	−2.175	−1.404	1.293	4.250	5.399	2.927	−4.223
0.45	16.076	5.815	0.777	−1.735	−2.409	−0.244	15.052
	−5.524	−2.143	1.110	4.608	7.281	8.475	−8.418

Table 2

ε_1	1	2	3	4	5	6	7	8
0.1	1.115	1.143	1.307	3.068	3.098	1.117	1.038	0.953
	0.046	0.048	0.026	−0.960	−1.029	0.048	0.058	0.161
0.2	1.600	1.568	1.298	3.313	3.503	1.521	1.176	0.821
	−0.001	−0.035	−0.195	−0.816	−1.255	0.045	0.263	0.703
0.3	2.893	3.007	3.390	4.193	4.600	2.405	1.529	0.639
	−0.428	−0.470	−0.603	−0.831	−1.955	−0.074	0.770	1.945
0.4	6.875	6.984	7.329	7.967	8.146	4.873	2.658	0.432
	−1.880	−1.910	−2.004	−2.177	−4.222	−0.370	2,478	5.729
0.45	14.981	15.094	15.443	16.076	15.052	9.506	4.907	0.295
	−5.141	−5.179	−5.295	−5.497	−8.371	−0.666	6.093	13.291

Table 3

ε_1	1	2	3	4	5	6
0.1	0.196	−0.072	−0.007	0.007	0.006	0.003
	0.685	1.094	1.029	1.001	0.996	0.997
0.2	0.524	−0.229	−0.036	0.022	0.020	0.011
	0.134	1.284	1.118	1.008	0.986	0.988
0.3	0.649	−0.346	−0.110	0.035	0.039	0.022
	−0.151	1.384	1.287	1.032	0.973	0.975
0.4	0.571	−0.279	−0.247	0.031	0.058	0.034
	−0.158	1.184	1.558	1.095	0.964	0.959
0.45	0.569	−0.143	−0.337	0.012	0.062	0.038
	−0.201	0.916	1.728	1.157	0.969	0.953

segment OA (point 1 −4) and a step $2(c - R)/3$ on the segment CD (points 5 −8).

Table 3 contains the stresses $Y_y^{(1)}$ along the horizontal symmetry axis. The points are sampled beginning from the coordinate origin, in the positive direction of the x-axis, using a step of $0.4\,a$.

The stresses at the points belonging to the longitudinal boundaries were computed using a step of $0.3\,a$ for all given values of ε_1. The values obtained for the stresses $X_y^{(1)}$ and $Y_y^{(1)}$ were used to draw conclusions about the accuracy with which the boundary conditions were satisfied. The stress $X_x^{(1)}$ itself is of particular interest. When $T_x = \text{const} \neq 0$, $T_y = 0$, $\varepsilon_1 = 0.45$, the stress varies at the points indicated (with the accuracy of up to the factor T_x) as follows: it attains a minimum of 0.2951 at the point D, is equal to 7.0142 at the point $x = 0.3\,a$, and from then on it diminishes in value and tends to unity ($X_x^{(1)} = 1.0109\,T_x$ at $x = 2.1\,a$).

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